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A CONTINUED FRACTION REPRESENTATION

FOR THEODORSEN'S CIRCULATION FUNCTION

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SUMMARY

A continued fraction representation for Theodorsen's circulation function is derived. This continued fraction converges to the circulation function everywhere except on the branch cut. It can be used to compute the function except when the argument is small. When converted to pole-residue form the continued fraction greatly facilitates the evaluation of integrals containing the circulation function.

INTRODUCTION

Theordorsen's circulation function (ref. 1) relates lift to downwash in unsteady incompressible potential flow. The function can be expressed as the ratio of two continguous confluent hypergeometric functions and hence has a continued fraction representation derivable from the continued fraction of Gauss.

This continued fraction can be truncated to give rational approximations to the circulation function. These approximations are useful in control theory because their poles and zeroes are easily computed. These approximations can also be inverse Laplace transformed to give accurate approximations to Wagner's function.

SYMBOLS

A	recursion coefficient matrix
A_n	truncated continued fraction numerator
$A_{\ell k}$, B_ℓ	coefficients of least squares simultaneous equations
a_n	continued fraction coefficients (numerator)
a_k, b_k, c_k	polynomial recursion coefficients
B_n	truncated continued fraction denominator
b_n	continued fraction coefficients (denominator)
$C(-is)$	Theodorsen's circulation function
$C_n(-is)$	truncated continued fraction approximation to $C(-is)$
$\bar{C}(\omega)$	least square approximation to $C(\omega)$

d_k	diagonal elements of A-matrix
$E_{2n}(t)$	correction integral used when evaluating Wagner's function
e_k	subdiagonal and superdiagonal A-matrix elements
${}_2F_o(a, b;;z)$	confluent hypergeometric function
$F(-is)$	even part of $C(-is)$ (real part of $C(-is)$ if $-is$ is real)
$G(-is)$	$[C(-is) - F(-is)]/(i2)$
I_ν	modified Bessel function of the first kind
i	$\sqrt{-1}$
J_ν, Y_ν	Bessel function of first and second kind
n	truncation order
n'	number of residues that contribute significantly to $C_{2n}(-is)$
P_n	even truncation numerator
$\overline{P}_n$	odd truncation numerator
Q_n	even truncation denominator
$\overline{Q}_n$	odd truncation denominator
R	ratio of two contiguous confluent hypergeometric functions
$R_k(x)$	any polynomial of degree k satisfying a three term recursion relation
s	complex argument of circulation function = $\sigma + i\omega$
s_k	poles of $C_{2n}(-is)$
s_k'	zeros of numerator in $C_{2n}(-is)$
t	time
X	polynomial column vector
x	$4s$
Δt	result spacing in FFT quadrature
u_k, v_k	coefficients used in least squares approximation

$\phi(t)$	Wagner's function
ω_o	a value of ω such that $\lvert C_{2n}(\omega) - C(\omega)\rvert < \varepsilon$ if $\omega > \omega_o$
$\Delta\omega$	step size used in FFT quadrature
$\sim$	asymptotic to
$\approx$	approximately equal
λ	eigenvalue of A (= $-4s$)
ν	arbitrary order of Bessel function
σ	real part of s
ω	imaginary part of s

THE CONTINUED FRACTION

Theodorsen's circulation function can be expressed

$$C(-is) = K_1(s)/(K_o(s) + K_1(s)) \tag{1}$$

The region of aerodynamic interest in the complex s-plane lies on or near the positive imaginary axis. The Bessel function $K_\nu(s)$ is expressible as a confluent hypergeometric function (eqs. 13.6.21 and 13.1.10.2 of ref. 2).

$$K_\nu(s) = \left(\frac{\pi}{2s}\right)^{\frac{1}{2}} e^{-s} \, {}_2F_o\left(\tfrac{1}{2} + \nu, \tfrac{1}{2} - \nu;; -\frac{1}{2s}\right) \tag{2}$$

Replacing the Bessel functions in $C = 1 - K_o/(K_o + K_1)$ by confluent hypergeometric functions and using Gauss' relation for contiguous functions (eq. 15.2.14 or 13.4.17-18 of ref. 2) to combine the numerator functions gives

$$C(-is) = 1 - \tfrac{1}{2} \cdot {}_2F_o\left(\tfrac{1}{2}, \tfrac{1}{2};; -\frac{1}{2s}\right)/{}_2F_o\left(\tfrac{1}{2}, -\tfrac{1}{2};; -\frac{1}{2s}\right) \tag{3}$$

The ratio of two contiguous confluent hypergoemetric functions

$$R = {}_2F_o\,(a, b;; z)/{}_2F_o\,(a, b - 1;; z) \tag{4}$$

has a very simple continued fraction representation (the confluent form of the continued fraction of Gauss, see chapter XVIII of ref. 3). It is

$$R = \cfrac{1}{1 - \cfrac{az}{1 - \cfrac{bz}{1 - \cfrac{(a+1)z}{1 - \cfrac{(b+1)z}{1 - \cfrac{(a+2)z}{1 - \cdots}}}}}} \tag{5}$$

so

$$C(-is) = b_o + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \tag{6}$$

where $b_o = 1, \quad a_1 = -\frac{1}{2}, \quad b_1 = 1$ (7)

and

$$\begin{aligned} a_{2n} &= 2n - 1, \quad b_{2n} = 4s \\ & \qquad\qquad\qquad\qquad n = 1,2,\ldots \\ a_{2n+1} &= 2n - 1, \quad b_{2n+1} = 1 \end{aligned} \tag{8}$$

That is,

$$C(-is) = 1 - \cfrac{\frac{1}{2}}{1 + \cfrac{1}{4s + \cfrac{1}{1 + \cfrac{3}{4s + \cdots}}}} \tag{9}$$

This continued fraction converges to $C(-is)$ in the entire complex s-plane cut on the negative real axis.

RATIONAL APPROXIMATIONS

Let $C_n(-is)$ represent the continued fraction of equation (6) and (7) truncated by discarding all terms beyond a_n/b_n. That is, by setting a_{n+1} to zero. This can be expressed as a rational function

$$C_n = A_n/B_n \tag{10}$$

where A_n, B_n are polynomials in $4s$ computed by the usual forward recursion formula (ref. 3) for continued fractions, namely

$$A_{-1} = 1; \quad B_{-1} = 0; \quad A_o = b_o; \quad B_o = 1$$

$$A_k = b_k A_{k-1} + a_k A_{k-2}; \quad B_k = b_k B_{k-1} + a_k B_{k-2}, \; k = 1,2,\ldots \tag{11}$$

Since the even and odd terms of the fraction have different forms (eq. (8)), it is convenient to separate the even and odd subscripts in recursion (11). Let

$$C_{2n}(-is) = \tfrac{1}{2} P_n(4s)/Q_n(4s) \tag{12}$$

$$C_{2n+1}(-is) = \tfrac{1}{2} \bar{P}_n(4s)/\bar{Q}_n(4s) \tag{13}$$

where

$$P_n = 2A_{2n}, \quad Q_n = B_{2n}$$

$$\bar{P}_n = 2A_{2n+1}; \quad \bar{Q}_n = B_{2n+1} \tag{14}$$

The recursion formulas for P_n, Q_n, $\bar{P}_n$ and $\bar{Q}_n$ are derived from equation (11) and are

$$P_o(x) = 2; \qquad Q_o(x) = 1$$

$$P_1(x) = x + 2; \qquad Q_1(x) = x + 1 \tag{15}$$

$$P_{k+1}(x) = (x + 4k)P_k(x) - (2k - 1)^2 P_{k-1}(x)$$

$$Q_{k+1}(x) = (x + 4k)Q_k(x) - (2k - 1)^2 Q_{k-1}(x) \tag{16}$$

$$\bar{P}_o(x) = 1 \qquad \bar{Q}_o(x) = 1$$

$$\bar{P}_1(x) = x + 3 \qquad \bar{Q}_1(x) = x + 2 \tag{17}$$

$$\bar{P}_{k+1}(x) = (x + 4k + 2)\bar{P}_k(x) - (4k^2 - 1)\bar{P}_{k-1}(x)$$

$$\bar{Q}_{k+1}(x) = (x + 4k + 2)\bar{Q}_k(x) - (4k^2 - 1)\bar{P}_{k-1}(x) \tag{18}$$

It can be seen that P_n, Q_n, $\bar{P}_n$, and $\bar{Q}_n$ are all polynomials in $x = 4s$ of degree n for $n > 0$. The first few polynomials of each set are

$$Q_o = 1 \qquad Q_1 = x + 1$$

$$Q_2 = x^2 + 5x + 3; \qquad Q_3 = x^3 + 13x^2 + 34x + 15$$

$$Q_4 = x^4 + 25x^3 + 165x^2 + 298\,x + 105$$

$$Q_5 = x^5 + 41x^4 + 516x^3 + 2301x^2 + 3207x + 945 \tag{19}$$

$$P_o = 2; \qquad P_1 = x + 2$$

$$P_2 = x^2 + 6x + 6; \qquad P_3 = x^3 + 14x^2 + 45x + 30$$

$$P_4 = x^4 + 26x^3 + 188x^2 + 420x + 210$$

$$P_5 = x^5 + 42x^4 + 555x^3 + 2742x^4 + 4725x + 1890 \tag{20}$$

$$\bar{Q}_o = 1; \qquad \bar{Q}_1 = x + 2$$

$$\bar{Q}_2 = x^2 + 8x + 9; \qquad \bar{Q}_3 = x^3 + 18x^2 + 74x + 60$$

$$\bar{Q}_4 = x^4 + 32x^3 + 291x^2 + 216x + 525$$

$$\bar{Q}_5 = x^5 + 50x^4 + 804x^3 + 4920x^2 + 10551x + 5670 \tag{21}$$

$$\bar{P}_o = 1; \qquad \bar{P}_1 = x + 3$$

$$\bar{P}_2 = x^2 + 9x + 15; \qquad \bar{P}_3 = x^3 + 19x^2 + 90x + 105$$

$$\bar{P}_4 = x^4 + 33x^3 + 321x^3 + 1050x + 945$$

$$\bar{P}_5 = x^5 + 51x^4 + 852x^3 + 5631x^2 + 14175x + 10395 \tag{22}$$

Inspection of the first new polynomials and the recursion formulas shows that

$$Q_n(0) = (2n - 1)!!$$

$$P_n(0) = 2(2n - 1)!!$$

$$Q_n(0) = (n + 1)(2n - 1)!!$$

$$P_n(0) = (2n + 1)!! \tag{23}$$

$$C_{2n}(0) = 1$$

and

$$C_{2n+1}(0) = 1 - \frac{1}{2n + 2}$$

The even numbered convergents C_{2n} give the correct value of $C(0)$ while the odd convergents C_{2n+1} merely approach the correct value. Because of this, and because they have a slightly simpler eigenvalue matrix, the even convergents are much more convenient to use.

The even convergents are the diagonal elements, and the odd convergents are subdiagonal elements of a Padé matrix defined by setting its first column to the convergents of the asymptotic series for $C(-is)$ and setting its first row to those of the asymptotic series for $1/C(-is)$. Reference 4 tabulates the first few diagonal Padé elements. However, the expression for $C_8(-is)$ in reference 4 is incorrect. Fortunately reference 4 makes no further use of $C_8(-is)$.

POLES AND ZEROS

All the poles and zeros of $C_{2n}(-is)$ lie on the negative real axis. If they are numbered in order of distance from the origin they satisfy the inequality

$$-\infty < s'_n < s_n < s'_{n-1} \ldots < s'_2 < s_2 < s'_1 < s_1 < 0$$

where s_k are the poles (zeros of Q_n) and s'_k are the zeros (zeros of P_n).

It is not practical to compute the poles of $C_{2n}(-is)$ by solving $Q_n(x) = 0$ as a polynomial equation if n is large because $Q_n(x)$ overflows the computer if x is barely outside of the convex set containing all of the roots. Instead the recursion formula (16) is used to construct matrices whose eigenvalues are the poles and zeros of $C_{2n}(-is)$.

Suppose the polynomials $R_k(x)$ are each of degree k for $k \geq 0$ and satisfy the three term recursion relation

$$a_k R_{k+1} - (x + b_k) R_k + c_k R_{k-1} = 0 \qquad (24)$$

If $R_{-1} \neq 0$ then a_0 and b_0 should be redefined so R_{-1} does equal 0. If $R_n(x) = 0$, then equation (24) can be written in matrix form as

$$
\begin{bmatrix}
b_o & -a_o & 0 & 0 & 0 & & 0 \\
-c_1 & b_1 & -a_1 & 0 & 0 & & 0 \\
0 & -c_2 & b_2 & -a_3 & 0 & & 0 \\
0 & 0 & -c_3 & b_3 & & & 0 \\
 & & & & & . & 0 \\
 & & & & & & . \\
0 & 0 & 0 & 0 & 0 & -c_{n-1} & b_{n-1}
\end{bmatrix}
\begin{bmatrix} R_o \\ R_1 \\ R_2 \\ R_3 \\ \\ \\ R_{n-1} \end{bmatrix}
= -x
\begin{bmatrix} R_o \\ R_1 \\ R_2 \\ R_3 \\ \\ \\ R_{n-1} \end{bmatrix}
\tag{25}
$$

This is a matrix eigenvalue problem

$$AX = \lambda X \tag{26}$$

where $\lambda = -x$ (27)

$$X = \text{col } (R_o, R_1, \ldots R_{n-1}) \tag{28}$$

and A is a tridiagonal matrix whose kth row is

$$(0, \ldots, -c_{k-1}, b_{k-1}, -a_{k-1}, \ldots 0) \tag{29}$$

The matrix A is made symmetric by replacing R_k by $\gamma_k \overline{R}_k$ where

$$\gamma_k/\gamma_{k-1} = \sqrt{c_k/a_{k-1}} \tag{30}$$

Then $AX = \lambda X$ where now (31)

$$X = \text{col}\,(\bar{R}_o, \bar{R}_1, \ldots \bar{R}_{n-1}) \tag{32}$$

and the kth row of A is

$$(0, \ldots, e_k, d_k, e_{k+1}, \ldots 0) \tag{33}$$

where

$$d_k = b_{k-1} \tag{34}$$

as before and

$$e_k = \sqrt{c_{k-1} a_{k-2}} \tag{35}$$

To compute the poles s_k of $C_{2n}(-is)$ let $R_k = Q_k$. Then

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \\ -1 & 4 & -3 & 0 & 0 & \\ 0 & -3 & 8 & -5 & 0 & \\ 0 & 0 & -5 & 12 & -7 & \\ & & & & \cdot & \cdot \\ & & & & \cdot & \cdot \end{bmatrix} \tag{36}$$

and $s_k = -\tfrac{1}{4}\lambda_k$. (37)

That is

$$\left.\begin{array}{ll}
 & d_1 = 1 \\
 & e_2 = 1 \\
 & d_2 = 4 \\
\text{for} & k = 3 \text{ to } n \\
 & d_k = d_{k-1} + 4 \\
 & e_k = e_{k-1} - 2 \\
 & \text{next } k
\end{array}\right\} \tag{38}$$

Similarly the zeros s_k' of $C_{2n}(-is)$ are obtained by letting $R_k = P_k$. Then

$$A = \begin{bmatrix} 2 & -\sqrt{2} & 0 & 0 & 0 & \\ -\sqrt{2} & 4 & -3 & 0 & 0 & \\ 0 & -3 & 8 & -5 & 0 & \\ 0 & 0 & -5 & 12 & -7 & \\ & & & & & \cdot \\ & & & & & \cdot \end{bmatrix} \tag{39}$$

$$s_k' = -\tfrac{1}{4}\lambda_k \tag{40}$$

That is

$$
\left.
\begin{array}{ll}
 & d_1 = 2 \\
 & e_2 = \sqrt{2} \\
 & d_2 = 4 \\
 & e_3 = -3 \\
 & d_3 = 8 \\
\text{for} & k = 4 \text{ to } n \\
 & d_k = d_{k-1} + 4 \\
 & e_k = e_{k-1} - 2 \\
 & \text{next } k
\end{array}
\right\}
\qquad (41)
$$

Both A matrices are tridiagonal, symmetric, and positive definite. Eigenvalues can easily be computed even for very large n because the nonzero matrix elements are easy to compute and do not vary widely in magnitude. Table I was computed using procedure tql1 on page 232 of reference 5. It lists the poles s_k and zeros s_k' of $C_{2n}(-is)$ for $n = 1, 2, 4,$ and 8.

POLES AND RESIDUES

The expression for $C_{2n}(-is)$ can be written

$$C_{2n}(-is) = \frac{1}{2} + \frac{P_n(4s) - Q_n(4s)}{2Q_n(4s)} \qquad (42)$$

$P_n - Q_n$ is of lower degree than Q_n and the zeros of Q_n are all distinct so, using partial fractions, one obtains

$$C_{2n}(-is) = \frac{1}{2} + \sum_{k=1}^{n} \frac{r_k}{s - s_k} \qquad (43)$$

The coefficients of the partial fraction r_k are the residues of the poles s_k. They can be computed from either

$$r_k = \frac{P_n(4s_k)}{8Q_n'(4s_k)} \tag{44}$$

or

$$r_k = \frac{1}{2}(s_k - s_k') \prod_{\substack{\ell=1 \\ \ell \neq k}}^{n} \frac{s_k - s_\ell'}{s_k - s_\ell} \tag{45}$$

The residues approach zero rapidly as k increases. This is because of the factor $s_k - s_k'$ in equation (45). Except for the upper left hand corner elements d_1 and e_2 the eigenvalue matrices for s_k and s_k' are identical. These corner elements have little effect on the higher eigenvalues because the matrices are diagonally dominated. The fact that $r_k \to 0$ rapidly as k increases is important because it means that when n is large the sum (43) can be truncated as n' where $n' < n$ and

$$|r_k/s_k| < \varepsilon \tag{46}$$

for all $k > n'$. Then

$$C_{2n}(-is) \approx \frac{1}{2} + \sum_{k=1}^{n'} \frac{r_k}{s - s_k} \tag{47}$$

The number of terms in the sum (47) increases much more slowly than n. Table II, which is a continuation of table I, lists the poles, zeros, and residues for $n = 16$, 32, 64 and 128. It also lists n' based on $\varepsilon = 10^{-10}$. Note how slowly n' increases with n.

APPLICATIONS

Applications of the continued fraction representation of $C(-is)$ include its use to evaluate the function and its use to represent or to evaluate integrals (particularly infinite limit integrals) containing $C(-is)$.

The error contours of $C_{2n}(-is)$ resemble a family of parabolas containing the negative real axis and with a common focus at the origin. That is, if $s = \sigma + i\omega$ (σ,ω real), then the error contours are approximated by the family of parabolas

$$2\omega_o\sigma + \omega^2 = \omega_o^2 \tag{48}$$

ω_o, the intercept of the parabola with the ω-axis, is a function of ε, the error tolerance, and of n. Given n and ε if ω_o has been computed then $|C_{2n}(-is) - C(-is)| < \varepsilon$ for all $s = \sigma + i\omega$ for which

$$\omega^2 + 2\omega_o\sigma > \omega_o^2 \tag{49}$$

It is hard to compute ω_o from ε and n so the usual procedure is to choose ω_o and then compute

$$\varepsilon = |C_{2n}(\omega_o) - C(\omega_o)| \tag{50}$$

For example, if $\omega_o = 2$ and $n = 8$, then $\varepsilon = 0.3 \times 10^{-12}$ and the error in C_{2n} is less than ε for all real $\omega > 2$ and for all complex $s = \sigma + i\omega$ for which $4\sigma + \omega^2 > 4$.

The continued fraction is the most efficient way available to compute $C(\omega)$ if $\omega \geq 2$ or if complex $s = \sigma + i\omega$ satisfies $4\sigma + \omega^2 \geq 4$. It should be used in pole-residue form truncated at n' for small ω. If $\omega > 20$ (or if $400\sigma + \omega^2 > 400$) it should be used as a truncated asymptotic series. The asymptotic series for $C(-is)$ is

$$C(-is) \sim \frac{1}{2}\left[1 - \frac{1}{4s} - \frac{2}{(4s)^2} + \frac{7}{(4s)^3} - \frac{38}{(4s)^4} + \frac{286}{(4s)^5} - \frac{2756}{(4s)^6} + \frac{32299}{(4s)^7} - \cdots\right] \tag{51}$$

as $s \to \infty$ for $|\arg(s)| < \frac{3\pi}{2}$.

The asymptotic series is obtained from equation (9) by repeated division. The series (51) diverges for all s. The asymptotic series, unlike the continued fraction, can be used to approximate $C(-is)$ on the branch cut, $\arg(s) = \pm\pi$, if $|s|$ is sufficiently large.

The other use of the continued fraction mentioned at the beginning of this section is to facilitate evaluation of integrals containing $C(-is)$. For this application the pole-residue form, equation (47), is used. Two examples are presented. One is the evaluation of Wagner's function $\phi(t)$. The other is the evaluation of some integrals that occur when approximating $C(-is)$ using least squares.

Wagner's Function

Wagner's function $\phi(t)$, is the inverse Laplace transform of $C(-is)/s$.

$$\phi(t) = \frac{1}{i2\pi} \int_{c-i\infty}^{c+i\infty} e^{st} \frac{C(-is)}{s} \, ds \tag{52}$$

Each rational approximation to $C(-is)$

$$C_{2n}(-is) = \frac{1}{2} + \sum_{k=1}^{n' \leq n} \frac{r_k}{s - s_k} \tag{53}$$

has an associated exponential approximation to $\phi(t)$

$$\phi_{2n}(t) = 1 + \sum_{k=1}^{n' \leq n} \frac{r_k}{s_k} e^{s_k t} \tag{54}$$

obtained by substituting equation (53) into equation (52) and performing the indicated integral transformation. The 1 appearing in equation (54) is computed from

$$C_{2n}(0) = \frac{1}{2} + \sum_{k=1}^{n} \frac{r_k}{s_k} = 1 \tag{55}$$

This is only true for the diagonal Padé elements C_{2n}. For the subdiagonal Padé elements

$$C_{2n+1}(0) = \frac{1}{2} + \sum_{k=1}^{n} \frac{\bar{r}_k}{s_k} = 1 - \frac{1}{2n + 2} \tag{56}$$

$$\phi_{2n+1}(t) = 1 - \frac{1}{2n + 2} + \sum_{k=1}^{n' \leq n} \frac{\bar{r}_k}{s_k} e^{\bar{s}_k t} \tag{57}$$

The odd approximants to $\phi(t)$ do not give the correct limit at $t=\infty$. The approximation (54) can be used to compute $\phi(t)$ accurately, even for small n, if t is sufficiently small. For large t equation (54) has too strong an exponential decay.

Equation (54) illustrates the use of the pole-residue form of the continued fraction for $C(-is)$ to replace a numerical integration by a closed form integration. It can also be used to simplify a numerical integration. To integrate equation (52) the path of integration $(c-i\infty,\ c+i\infty)$ must be fixed. The only restriction is that the path be to the right of the branch point at $s=0$ and have a nonpositive real part at the two ends. Two paths are very convenient for numerical integration. One is the imaginary axis as shown in figure 1. The other is the branch cut as shown in figure 2.

If path 1 is used and symmetry of the integrand is considered (see sections 5 through 7 of ref. 6 for details) one obtains the following integral representation for $\phi(t)$

$$\phi(t) = \frac{2}{\pi} \int_0^\infty \frac{\sin \omega t}{\omega} F(\omega)\, d\omega \tag{58}$$

If path 2 is used and symmetry of the integrand is considered one obtains

$$\phi(t) = 1 - \int_0^\infty \frac{s^{-2} e^{-st} ds}{(K_o - K_1)^2 + \pi^2(I_o + I_1)^2} \tag{59}$$

The continued fraction for $C(-is)$ cannot be used to help evaluate equation (59) because the continued fraction diverges along the branch cut. However, the continued fraction can provide considerable help in evaluating equation (58) as will be shown.

The function $F(\omega)$ appearing in equation (58) is usually defined to be the real part of $C(\omega)$. However, if complex arguments are permitted, it is more convenient to define it to be the even part of $C(-is)$.

$$F(-is) = \frac{1}{2}[C(-is) + C(is)] \tag{60}$$

The function $F(-is)$ usually occurs in association with the function $G(-is)$ defined

$$G(-is) = \frac{1}{i2}[C(-is) - C(is)] \tag{61}$$

Equation (53) furnishes rational approximations to F and G

$$F_{2n}(\omega) = \frac{1}{2} + \sum_{k=1}^{n' \leq n} \frac{-s_k r_k}{s_k^2 + \omega^2} \tag{62}$$

$$G_{2n}(\omega) = \omega \sum_{k=1}^{n' \leq n} \frac{r_k}{s_k^2 + \omega^2} \tag{63}$$

If the $F(\omega)$ in equation (58) is expressed

$$F(\omega) = F_{2n}(\omega) + [F(\omega) - F_{2n}(\omega)] \tag{64}$$

and the first F_{2n} is integrated in closed form one obtains

$$\phi(t) = \phi_{2n}(t) + E_{2n}(t) \tag{65}$$

where $\phi_{2n}(t)$ is given by equation (54) and the error or correction term $E_{2n}(t)$ is

$$E_{2n}(t) = \frac{2}{\pi} \int_0^{\omega_0} \frac{\sin \omega t}{\omega} [F(\omega) - F_{2n}(\omega)] \, d\omega \tag{66}$$

Equation (66) is much easier to integrate numerically than equation (58). This is because it has finite limits and because a large relative error can be tolerated. Infinite limit oscillatory integrals are notoriously difficult to compute. Equation (66) has a finite upper limit because $F - F_{2n}$ is essentially zero for $\omega \geq \omega_o$. A large relative error can be tolerated because $|F - F_{2n}| \ll F$ even for $\omega < \omega_o$.

If equation (66) is to be integrated for a single value of t a sophisticated integration technique such as Legendre-Gauss or Romberg quadrature can be used. However, if a large number of values of t are used, then equation (66) should be evaluated as a trapezoidal sum using a fast Fourier transform. For a quadrature order m let

$$\Delta\omega = \frac{\omega_o}{m}$$

$$\omega = k\Delta\omega, \qquad k = 0 \text{ to } m \tag{67}$$

$$t = t_o + \ell\Delta t, \qquad \ell = 0 \text{ to } m$$

The FFT formalism requires that

$$\Delta\omega\Delta t = 2\pi/m \tag{68}$$

so

$$\Delta t = \frac{2\pi}{\omega_o} \tag{69}$$

The trapezoidal sum for E_{2n} is

$$E_{2n}(t_o + \ell\Delta t) \approx \frac{2}{\pi}\,\Delta\omega \sum_{k=0}^{m}{}'' \sin k\Delta\omega\,(t_o + \ell\Delta t) \cdot \frac{F(k\Delta\omega) - F_{2n}(k\Delta\omega)}{k\Delta\omega} \tag{70}$$

or

$$E_{2n}\left(t_o + \frac{2\pi}{\omega_o}\ell\right) \approx -\ell m \left\{ \frac{2}{\pi} \sum_{k=0}^{m-1} e^{-i2\pi \ell k/m} f_k \right\} \tag{71}$$

for $\ell = 0$ to $m - 1$

where $f_o = 0$ and

$$f_k = \frac{e^{-i \frac{\omega_o t_o}{m} k}}{k} \left[F\left(\frac{\omega_o}{m} k\right) - F_{2n}\left(\frac{\omega_o}{m} k\right) \right] \tag{72}$$

for $k = 1$ to $m - 1$.

By using a fast Fourier transform to evaluate the sum in equation (71), it is possible to evaluate $E_{2n}(t_o + 2\pi/\omega_o\, \ell)$ for m values of ℓ using only $\log_2(m)$ times as much computing effort as would be required for one value of ℓ. However, only the $E_{2n}(t_o + 2\pi/\omega_o\, \ell)$ for $\ell < m/4$ are reasonable approximations to the integral (66). The term t_o in the argument of E_{2n} is to permit interpolating between values of $\ell\Delta t$. It should not be larger than $2\pi/\omega_o$ or aliasing can occur. Thus, the largest value of t for which $\phi(t)$ can be computed is

$$t = \frac{\pi m}{2\omega_o} \tag{73}$$

This can be increased either by increasing the quadrature order m or by increasing the exponential approximation order n, thereby decreasing ω_o.

The circulation function $F(\omega)$ in equation (72) can be computed using Bessel functions

$$F(\omega) = \frac{J_1(J_1 + Y_o) + Y_1(Y_1 - J_o)}{(J_1 + Y_o)^2 + (Y_1 - J_o)^2} \tag{74}$$

A very convenient way to compute J_ν or Y_ν for a large number of equispaced arguments

$$\omega = \frac{\omega_o}{m} k, \qquad k = 1 \text{ to } m - 1 \tag{75}$$

is to evaluate J_ν and Y_ν accurately at the two ends, $\omega = \omega_o/m$ and $\omega = \omega_o - \omega_o/m$ and then to approximate J_ν, Y_ν at all the intermediate points $(k = 2$ to $m - 2)$ by solving the Bessel equation as a finite difference boundary value problem. This permits computing J_ν, Y_ν at the intermediate points with less computing effort than is required for an elementary function such as a square root. It has the additional advantage that the accuracy with which the intermediate Bessel functions are computed increases as the quadrature order m is increased.

Equations (65), (53), and (71) can be used to compute $\phi(t)$ for any reasonable value of t. For unreasonable values of t such as 10^6 or to estimate the behavior of $\phi(t)$ as $t \to \infty$ the other integral representation, equation (59), should be used. It can be evaluated as a Laguerre-Gauss quadrature. When setting up a Laguerre-Gauss quadrature the exponential weight function should represent the behavior of the integrand at infinity rather than merely being an easily identifiable exponential factor. That is, in equation (59) the exponential weight function is $e^{-(2+t)s}$, not e^{-st}. The factor e^{-2s} comes from the asymptotic behavior of I_o and I_1. Thus

$$\phi(t) = 1 - \int_o^\infty e^{-(2+t)s} f(s)\, ds \tag{76}$$

where

$$f(s) = \frac{e^{2s}}{s^2[(K_o - K_1)^2 + \pi^2(I_o + I_1)^2]} \tag{77}$$

Letting $s = x/(2 + t)$ gives

$$\phi(t) = 1 - \frac{1}{2 + t} \int_o^\infty e^{-x} f\left(\frac{x}{2 + t}\right) dx \tag{78}$$

which can be integrated numerically to give

$$\phi(t) \approx 1 - \frac{1}{2+t} \sum_{k=1}^{m} w_k \, f\left(\frac{x_k}{2+t}\right) \tag{79}$$

where x_k, w_k are the Laguerre-Gauss abscissas and weights. As $t \to \infty$ the integral in (78) approaches

$$\int_0^\infty e^{-x} f(0) \, dx = 1 \tag{80}$$

so a very crude approximation to $\phi(t)$ is

$$\phi(t) \approx 1 - \frac{1}{2+t} \tag{81}$$

A slightly better approximation is obtained by retaining the next term in the expansion of $f(s)$ about the origin and integrating in closed form. This gives

$$\phi(t) \approx 1 - \frac{1}{2+t} \left[1 + \frac{2}{2+t} \ln (4 + 2t)\right] \tag{82}$$

Equations (81) or (82) show that $\phi(t)$ approaches 1 like $1/t$ as t increases rather than exponentially as indicated in equation (54).

A Least Squares Approximation to $C(-is)$

This section describes the use of the pole-residue form of the continued fraction for $C(-is)$ to evaluate some integrals that occur when generating a least squares rational approximation to $C(-is)$.

The continued fraction for $C(-is)$ is an expansion about $s = \infty + i0$ and hence is very accurate for large σ and $|\omega|$ (where $s = \sigma + i\omega$) and very uneconomical when σ and ω are both small. It has been shown (refs. 4 and 7) that economical rational approximations valid for $0 \leq \omega \leq \infty$ and $\sigma \approx 0$ can be generated by least squares.

There are several ways of generating a least squares rational approximation to $C(-is)$. The method described here has a moderately complicated derivation but is computationally very simple.

Let

$$\overline{C}(\omega) = \frac{1}{2} + \sum_{k+1}^{m} \frac{u_k}{v_k + i\omega} \tag{83}$$

where the constants u_k and v_k are chosen so $\overline{C}(\omega)$ approximates $C(\omega)$ over the entire positive real ω-axis. One way of computing the constants u_k and v_k is to minimize the error

$$E = \int_0^{\infty} \omega^{-\frac{1}{2}} \left|\overline{C}(\omega) - C(\omega)\right|^2 d\omega \tag{84}$$

The expression for $\overline{C}(\omega)$, equation (83), is linear in the u_k and the nonlinear in the v_k. Hence, it is very easy to minimize E with respect to the u_k and very hard to minimize E with respect to the v_k. For this reason the v_k are preassigned and the minimization is performed with respect to the u_k only. The v_k are chosen to be $-s_k$ for $n' = m$ if no better choice is available. The error expression E, equation (84), contains a weight function $\omega^{-\frac{1}{2}}$. This is included to origin weight the error. It is needed because $C(\omega)$ has a logarithmic branch point at $\omega = 0$ and hence is hard to approximate for small ω. All that has to be done to compute the u_k is to set $\partial E/\partial u_\ell$ to zero and solve the resulting system of linear simultaneous equations. This is performed as follows.

Let

$$\overline{C}(\omega) = \overline{F}(\omega) + i\overline{G}(\omega) \tag{85}$$

where

$$\overline{F}(\omega) = \frac{1}{2} + \sum_{k=1}^{m} \frac{u_k v_k}{v_k^2 + \omega^2} \tag{86}$$

$$\overline{G}(\omega) = -\omega \sum_{k=1}^{m} \frac{u_k}{v_k^2 + \omega^2} \tag{87}$$

Then

$$E = \int_0^\infty \omega^{-\frac{1}{2}} \left[(\overline{F} - F)^2 + (\overline{G} - G)^2 \right] \; d\omega \tag{83}$$

and

$$\frac{1}{2} \frac{\partial E}{\partial u_\ell} = \int_0^\infty \omega^{-\frac{1}{2}} \left[(\overline{F} - F) \frac{\partial \overline{F}}{\partial u_\ell} + (\overline{G} - G) \frac{\partial \overline{G}}{\partial u_\ell} \right] d\omega = 0$$

$$\text{for} \quad \ell = 1 \text{ to } m \tag{89}$$

This gives the system of equation

$$\sum_{k=1}^{m} A_{\ell k} \, u_k = B_\ell \qquad (\ell = 1 \text{ to } m) \tag{90}$$

where

$$A_{\ell k} = \frac{1}{\pi\sqrt{2}} \int_0^\infty \omega^{-\frac{1}{2}} \frac{v_\ell \, v_k + \omega^2}{(v_\ell^{\,2} + \omega^2)(v_k^{\,2} + \omega^2)} \; d\omega \tag{91}$$

$$B_\ell = \frac{1}{\pi\sqrt{2}} \int_0^\infty \omega^{-\frac{1}{2}} \frac{v_\ell (F(\omega) - \frac{1}{2}) - \omega G(\omega)}{v_\ell^{\,2} + \omega^2} \; d\omega \tag{92}$$

The factor $1/(\pi\sqrt{2})$ was appended to both $A_{\ell k}$ and B_ℓ because it makes some subsequent arithmetic easier.

The integral $A_{\ell k}$ can be evaluated in closed form. It is

$$A_{\ell k} = \frac{1}{v_\ell + v_k} \cdot \left(\frac{1}{\sqrt{v_\ell}} + \frac{1}{\sqrt{v_k}} \right) \tag{93}$$

The integral B_ℓ has to be computed numerically. A very easy way to perform this numerical integration is to replace $C = F + iG$ by C_{2n}, the pole-residue form of a truncated continued fraction, with n chosen large enough so that the error $C_{2n}(\omega) - C(\omega)$ is negligible compared to the error $\bar{C}(\omega) - C(\omega)$. Then

$$B_\ell = \sum_{k=1}^{n' \leq n} \frac{r_k}{v_\ell - s_k} \left(\frac{1}{\sqrt{v_\ell}} + \frac{1}{\sqrt{-s_k}} \right) \tag{94}$$

After $A_{\ell k}$ and B_ℓ have been computed, equation (90) can be solved for u_k.

The above procedure minimized E with respect to u_k for preassigned v_k. If one wants to minimize with respect to v_k, then substitute equation (90) into (88) to get the penalty function

$$E = E_o - \pi\sqrt{2} \sum_{k=1}^{m} B_k u_k \tag{95}$$

and perform a nonlinear optimization using some technique such as the Davidon-Fletcher-Powell algorithm (ref. 8). The constant E_o in equation (95) is given by

$$E_o = \pi\sqrt{2} \int_0^\infty \omega^{-\frac{1}{2}} \left[\left(F - \frac{1}{2} \right)^2 + G^2 \right] d\omega$$

It can be computed using $C_{2n}(-is)$ the same way B_ℓ was computed. However, since it is a constant, it is not needed for the optimization and can be set to zero in equation (95).

CONCLUDING REMARKS

The principal result of the investigation is the fact that Theodorsen's circulation function has a continued fraction representation with a particularly simple coefficient pattern, namely the consecutive odd integers. Although this continued fraction converges extremely slowly it still furnishes an economical way to compute the circulation function. The reason for this is that when converted to pole-residue form the terms containing distant poles can be discarded. For example, retaining only the 120 poles nearest to the origin from a 2048 term pole-residue form of the continued fraction introduces an additional error of only 10^{-10} to the approximation to $C(-is)$.

This investigation also furnished some information about the singularities of the circulation function. The Bessel functions that comprise $C(-is)$, namely $K_0(s)$ and $K_1(s)$, have no singularities other than logarithmic branch points at $s = 0$ and $s = \infty$. The only singularities that $C(-is)$ can possess are these two branch points and, possibly, poles at the zeros of the denominator in $C(-is)$, namely $K_0(s) + K_1(s)$. Since $C_{2n}(-is)$ converges to $C(-is)$ everywhere except when $\arg(s) = \pm\pi$, these zeros, if they exist at all, must all lie on the negative real axis. On the negative real axis the Bessel functions K_0 and K_1 can be expressed as functions of positive real argument by analytic continuation (see eq. 9.6.31 of ref. 2). The real part of this analytic continuation is the same for all sheets of the Riemann surface and is $K_0(x) - K_1(x)$ where $x = -s$ is real and positive. Inspection of figure 9.8 of reference 2, and of the asymptotic expression for K_0 and K_1, shows that $K_0(x) - K_1(x)$ has no positive real zeros. Hence $C(-is)$ has no singularities whatsoever except for logarithmic branch points at $s = 0$ and $s = \infty$.

The continued fraction representation of $C(-is)$ furnishes a very convenient way to compute Wagner's function from its definition as an inverse Laplace transformation. This leads either to an exponential approximation to Wagner's function or to an exponential approximation with a numerically integrated correction term. The latter can be used to compute $\phi(t)$ (but not $1 - \phi(t)$) to full register accuracy.

Another application of the continued fraction that was discussed involved its use to evaluate some integrals that occur when approximating $C(-is)$ by least squares. Low order approximations obtained using least squares are much more accurate over the frequency range of aerodynamic interest than the same order truncations of the continued fraction. However, high order approximations (those with over twenty terms) should be obtained by truncating the continued fraction because of numerical problems associated with the least squares process.

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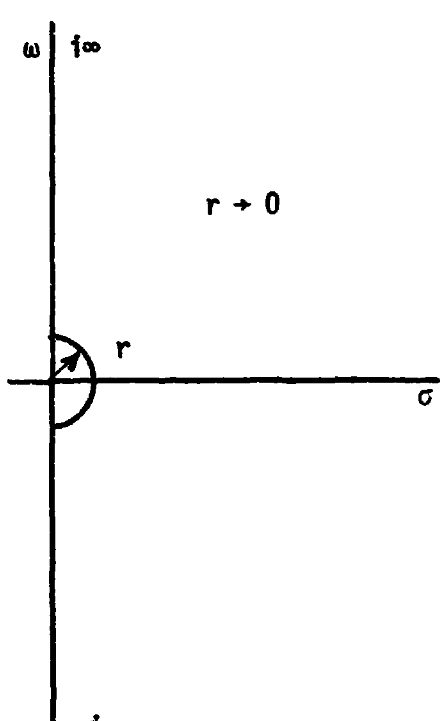

Figure 1
Imaginary Axis

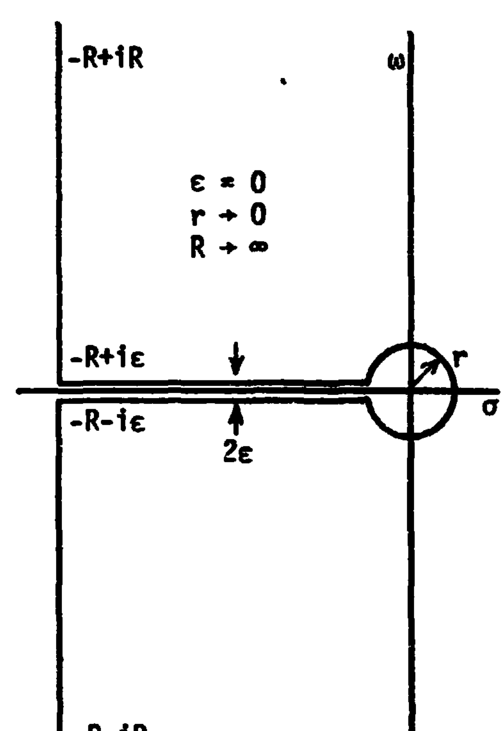

Figure 2
Branch Cut

TABLE I.

k	$-s_k$ poles	$-s_k'$ zeros	r_k residues
		n = 1	
1	0.2500000000	0.5000000000	0.1250000000
		n = 2	
1	0.1743060906	0.3169872981	0.0798343811
2	1.0756039094	1.1830127019	0.0451656189
		n = 4	
1	0.1156902163	0.1742812056	0.0414478723
2	0.5376897142	0.7031255226	0.0728261277
3	1.6861213269	1.7116460071	0.0105184411
4	3.9104987426	3.9109472647	0.0002075590
		n = 8	
1	0.0708292313	0.0895316590	0.0162792778
2	0.3014411613	0.4253867638	0.0677288394
3	0.8103741892	0.8990325533	0.0333426883
4	1.6931876604	1.7102975420	0.0069379972
5	2.9738355808	2.9753615342	0.0006840380
6	4.7259501453	4.7260072766	0.0000268518
7	7.1126863436	7.1126869821	0.0000003071
8	10.5616956880	10.5616956889	0.0000000005

TABLE II.

k	$-s_k$ poles	$-s_k'$ zeros	r_k residues
	$n = 16$	$n' = 12$	
1	.0399290405	.0450767705	.0049890236
2	.1770117985	.2309790353	.0410711992
3	.4213665832	.5226192420	.0460005529
4	.8319542602	.8955477962	.0226896684
5	1.4174350819	1.4379146184	.0078656754
6	2.1767569894	2.1814331978	.0019842800
7	3.1151752001	3.1159585681	.0003530589
8	4.2433929936	4.2434850891	.0000429141
9	5.5769738664	5.5769811202	.0000034466
10	7.1372634130	7.1372637778	.0000001755
11	8.9538356848	8.9538356959	.0000000054
12	11.0691194848	11.0691194850	.0000000001
	$n = 32$	$n' = 17$	
1	.0212539667	.0225775151	.0013308165
2	.1014880467	.1181364893	.0161066554
3	.2313703318	.2837315435	.0367590439
4	.4281625584	.5028891998	.0331916084
5	.7083237338	.7663017562	.0202106302
6	1.0728150641	1.1020615789	.0103265831
7	1.5200677586	1.5318978031	.0045717209
8	2.0496021188	2.0537719222	.0017407160
9	2.6619849235	2.6632677368	.0005642149
10	3.3585132712	3.3588528353	.0001544540
11	4.1409982347	4.1410745802	.0000354967
12	5.0116623678	5.0116768049	.0000068133
13	5.9731199112	5.9731221892	.0000010866
14	7.0284055993	7.0284058970	.0000001431
15	8.1810313620	8.1810313940	.0000000155
16	9.4350639127	9.4350639155	.0000000014
17	10.7952245954	10.7952245956	.0000000001
	$n = 64$	$n' = 23$	
1	.0109615336	.0112936532	.0003363445
2	.0550659069	.0594029977	.0045647906
3	.1282525325	.1451687092	.0165007629

TABLE II.- Continued.

	n = 64	n' = 23	
4	.2294179265	.2662436189	.0271832071
5	.3655073397	.4178090091	.0263132347
6	.5421118538	.5945791192	.0196294939
7	.7601447375	.7991787324	.0130424499
8	1.0190503483	1.0425922027	.0080538340
9	1.3182897476	1.3309553604	.0046499131
10	1.6575617149	1.6639218023	.0025025914
11	2.0367717604	2.0397792676	.0012503827
12	2.4559788735	2.4573135565	.0005779825
13	2.9153514543	2.9159042425	.0002465585
14	3.4151351711	3.4153478013	.0000968939
15	3.9556318607	3.9557075370	.0000350346
16	4.5371873692	4.5372122223	.0000116439
17	5.1601856915	5.1601932080	.0000035542
18	5.8250470008	5.8250490909	.0000009956
19	6.5322278278	6.5322283615	.0000002557
20	7.2822223566	7.2822224815	.0000000602
21	8.0755643170	8.0755643438	.0000000130
22	8.9128292551	8.9128292604	.0000000026
23	9.7946371142	9.7946371151	.0000000005
	n = 128	n' = 32	
1	.0055646485	.0056474386	.0000836954
2	.0286731117	.0297432178	.0011316178
3	.0686191075	.0729961100	.0047528127
4	.1238383560	.1351282785	.0115932142
5	.1941069901	.2155251600	.0183257845
6	.2813550805	.3130880575	.0204781205
7	.3878324001	.4262057173	.0183965411
8	.5145492100	.5533849095	.0147608539
9	.6616147368	.6949482493	.0111662180
10	.8288591342	.8538093471	.0081292943
11	1.0160927402	1.0331216673	.0057288341
12	1.2231683248	1.2341736061	.0039093147
13	1.4499857472	1.4568530792	.0025797170
14	1.6964847002	1.7006490926	.0016434498
15	1.9626365278	1.9650916323	.0010092418
16	2.2484371391	2.2498418722	.0005966957
17	2.5539011953	2.5546795382	.0003393272
18	2.8790574811	2.8794742689	.0001854763
19	3.2239453612	3.2241606823	.0000973962
20	3.5886121959	3.5887193728	.0000491158
21	3.9731115618	3.9731629082	.0000237800

TABLE II.- Concluded.

n = 128		n' = 32	
22	4.3775021010	4.3775257587	.0000110517
23	4.8018468266	4.8018573035	.0000049295
24	5.2462127401	5.2462171976	.0000021100
25	5.7106706464	5.7106724677	.0000003666
26	6.1952950885	6.1952958030	.0000003415
27	6.7001643509	6.7001646199	.0000001291
28	7.2253605012	7.2253605984	.0000000468
29	7.7709694533	7.7709694870	.0000000163
30	8.3370810429	8.3370810542	.0000000054
31	8.9237891120	8.9237891156	.0000000017
32	9.5311915985	9.5311915996	.0000000005